EE120 - Fall'19 - Lecture 18 Notes[1]

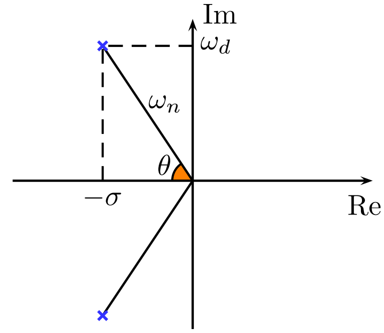
Murat Arcak

7 November 2019

Step Response of Second Order Systems

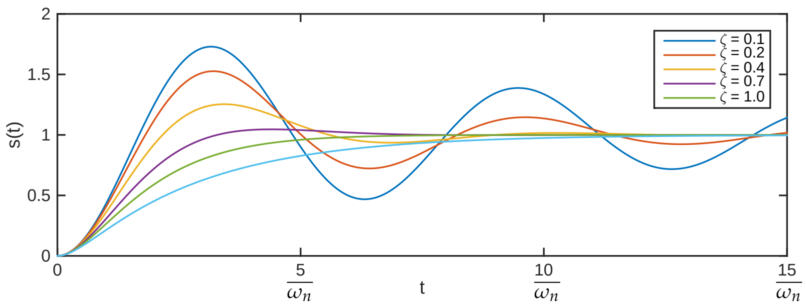
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ: damping ratio, ω_n: natural frequency



Poles: $s_{1,2} = \overbrace{-\omega_n cos\theta}^{\sigma} \mp \overbrace{j\omega_n sin\theta}^{\omega_d}$ where $\cos\theta = \zeta$

Below are the step responses for various values of ζ. Note that ω_n changes only the time scale, not the shape of the response.



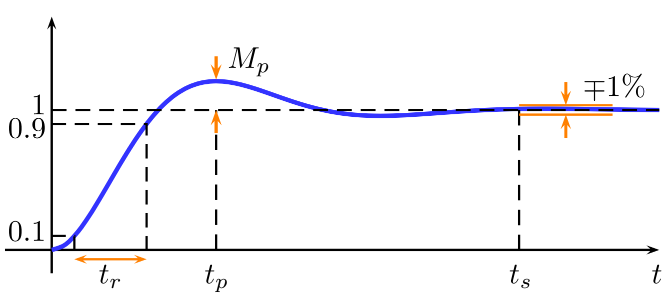
Important Features of the Step Response:

1) Rise time (tr): time to go from 10% to 90% of steady-state value

2) Peak overshoot (M_p): (peak value - steady state)/steady state

3) Peaking time (t_p): time to peak overshoot

4) Settling time (t_s): time after which the step response stays within 1% of the steady-state value



How do these parameters depend on ζ and ω_n?

$$u(t): \text{ unit step } \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

Step response:

$$\begin{aligned} Y(s) = \frac{1}{s}H(s) &= \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad (1) \\ &= \frac{A}{s} + \frac{B}{s+\sigma+j\omega_d} + \frac{B^*}{s+\sigma-j\omega_d} \\ & \quad A = 1 \quad B = -\frac{1}{2}\left(1+j\frac{\sigma}{\omega_d}\right) \end{aligned}$$

$$\begin{aligned} y(t) &= \left(1 + Be^{-\sigma t}e^{-j\omega_d t} + B^* e^{-\sigma t}e^{j\omega_d t}\right)u(t) \\ &= \left(1 + \underbrace{\left(Be^{-j\omega_d t} + B^* e^{j\omega_d t}\right)}_{= 2Re\{\underbrace{Be^{-j\omega_d t}}_{=-\frac{1}{2}\left(1+j\frac{\sigma}{\omega_d}\right)(cos\omega_d t - jsin\omega_d t)}\} = -\left(cos\omega_d t + \frac{\sigma}{\omega_d}sin\omega_d t\right)} e^{-\sigma t}\right)u(t) \end{aligned}$$

$$\boxed{y(t) = \left[1 - \left(cos\omega_d t + \frac{\sigma}{\omega_d}sin\omega_d t\right)e^{-\sigma t}\right]u(t)}$$

Peaking time:

$$\begin{aligned} \frac{d}{dt}y(t) &= \sigma e^{-\sigma t}\left(\cancel{cos\omega_d t} + \frac{\sigma}{\omega_d}sin\omega_d t\right) - e^{-\sigma t}\left(-\omega_d sin\omega_d t + \cancel{\sigma cos\omega_d t}\right) \\ &= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)sin\omega_d t \end{aligned}$$

$$\frac{d}{dt}y(t) = 0 \implies sin\omega_d t = 0 \qquad \boxed{t_p = \frac{\pi}{\omega_d}}$$

Peak overshoot: $M_p = y(t_p) - 1$

$$y(t_p) = \left(1 - \underbrace{cos\omega_d t_p}_{=cos\pi=-1} e^{-\sigma t_p}\right) = 1 + e^{-\sigma t_p} = 1 + e^{-\sigma\frac{\pi}{\omega_d}}$$

$$\boxed{M_p = e^{-\pi\frac{\sigma}{\omega_d}} = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}} \qquad \begin{array}{l} \zeta \nearrow \implies M_p \searrow \\ M_p \to 0 \text{ as } \zeta \to 1 \end{array} \qquad M_p \approx \begin{cases} 0.05 & \zeta = 0.7 \\ 0.16 & \zeta = 0.5 \end{cases}$$

Approximate expressions for rise time and settling time:

$$\boxed{t_s \approx \frac{4.6}{\sigma}} \qquad \text{(obtained from } e^{-\sigma t_s} = 0.01\text{)}$$

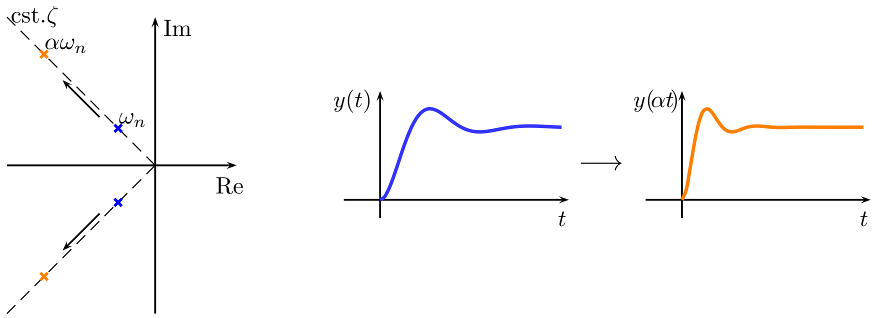
$$\boxed{t_r \approx \frac{1.8}{\omega_n}} \qquad \text{for } \zeta = 0.5 \text{ (changes little with } \zeta\text{)}$$

Note that t_p, t_s, t_r are inversely proportional to ω_n:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\underline{\omega_n}\sqrt{1-\zeta^2}} \quad t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\underline{\omega_n}\zeta} \quad t_r \approx \frac{1.8}{\underline{\omega_n}}.$$

This is consistent with our observation on page 1 that ω_n changes only the time scale, not the shape of the response. We make this property explicit in the following statement:

If ζ is kept constant and ω_n is scaled by a factor of $\alpha > 0$ ($\omega_n \to \alpha\omega_n$) then the step response is scaled in time by α: $y(t) \to y(\alpha t)$.



Proof: If we replace ω_n with $\alpha\omega_n$ in (1), we get

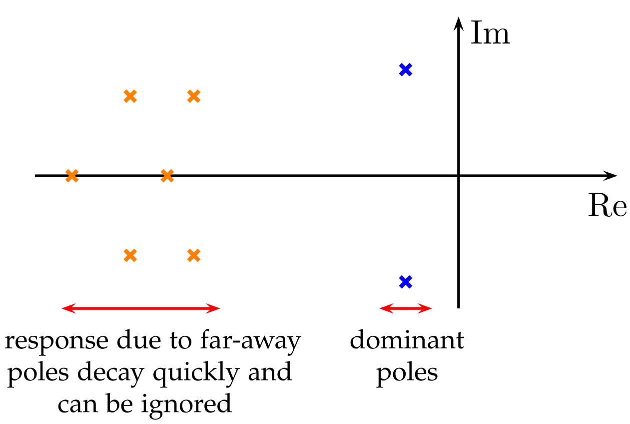
$$\frac{(\alpha\omega_n)^2}{s\left(s^2 + 2\zeta(\alpha\omega_n)s + (\alpha\omega_n)^2\right)} = \frac{\omega_n^2}{s\left(\left(\frac{s}{\alpha}\right)^2 + 2\zeta\omega_n\left(\frac{s}{\alpha}\right) + \omega_n^2\right)} = \frac{1}{\alpha}Y\left(\frac{s}{\alpha}\right).$$

The statement above then follows from the scaling property of Laplace transform:

$$y(\alpha t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{\alpha}Y\left(\frac{s}{\alpha}\right).$$

Summary: $\omega_n \nearrow$ increases speed of the response
$\zeta \nearrow$ reduces overshoot

Although the formulas above are for second order systems, they can be applied as approximate expressions to higher order systems with two dominant poles:



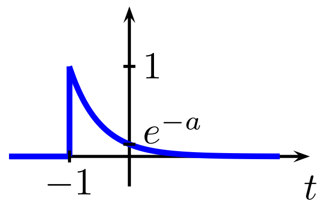
The Unilateral Laplace Transform

Section 9.9 *in* Oppenheim & Willsky

$$\boxed{\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt} \qquad (2)$$

Identical to the bilateral Laplace transform if $x(t) = 0$ for $t < 0$.

Example: $x(t) = e^{-a(t+1)}u(t+1)$



$$X(s) = \frac{e^s}{s+a} \qquad Re\{s\} > -a$$
$$\mathcal{X}(s) = \frac{e^{-a}}{s+a} \qquad Re\{s\} > -a$$

Properties of the Unilateral Laplace Transform

Most properties of the bilateral Laplace transform also hold for the unilateral Laplace transform.

Exceptions:

Convolution:

$$(x_1 * x_2)(t) \longleftrightarrow \mathcal{X}_1(s)\mathcal{X}_2(s) \;\; \text{if } x_1(t) = x_2(t) = 0 \text{ for all } t < 0$$

This follows from the convolution property of the bilateral Laplace transform which coincides with the unilateral transform because $x_1(t) = x_2(t) = 0,\ t < 0$.

Differentiation in Time:

$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow s\mathcal{X}(s) - x(0^-)}$$

Repeated application gives:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\} \longleftrightarrow s\left(s\mathcal{X}(s) - x(0^-)\right) - \frac{dx}{dt}(0^-)$$
$$= s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-)$$
$$\frac{d^3x(t)}{dt^3} = \frac{d}{dt}\left\{\frac{d^2x(t)}{dt^2}\right\} \longleftrightarrow s\left(s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-)\right) - \frac{d^2x}{dt^2}(0^-)$$
$$= s^3\mathcal{X}(s) - s^2x(0^-) - s\frac{dx}{dt}(0^-) - \frac{d^2x}{dt^2}(0^-)$$

Solving differential equations with the unilateral Laplace transform

<u>Example:</u>

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy}{dt} + 2y(t) = e^t \quad t \geq 0 \tag{3}$$

Initial condition $y(0^-) = a$, $\frac{dy}{dt}(0^-) = b$.

$$(s^2Y(s) - as - b) + 3(sY(s) - a) + 2Y(s) = \tfrac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = as + b + 3a + \tfrac{1}{s-1} = \tfrac{as^2 + (b+2a)s + (1-b-3a)}{s-1}$$

$$Y(s) = \tfrac{as^2 + (b+2a)s + (1-b-3a)}{(s+1)(s+2)(s-1)}$$

Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{B}{s-1} \\ &= \frac{(A_1 + A_2 + B)s^2 + (A_1 + 3B)s + (2B - 2A_1 - A_2)}{(s+1)(s+2)(s-1)} \end{aligned}$$

Match coefficients:

$$\left.\begin{aligned} A_1 + A_2 + B &= a \\ A_1 + 3B &= b + 2a \\ 2B - 2A_1 - A_2 &= 1 - b - 3b \end{aligned}\right\} \quad \begin{aligned} B &= 1/6 \\ A_1 &= -\tfrac{1}{2} + 2a + b \\ A_2 &= \tfrac{1}{3} - a - b \end{aligned}$$

Then,

$$y(t) = \frac{1}{6}e^t + \left(-\frac{1}{2} + 2a + b\right)e^{-t} + \left(\frac{1}{3} - a - b\right)e^{-2t} \quad t \geq 0.$$

Compare this to the standard method for solving linear constant coefficient differential equations:

The first term in $y(t)$ above is the particular solution. If we substitute $y_p(t) = \frac{1}{6}e^t$ in (3):

$$\frac{d^2y_p(t)}{dt^2} + 3\frac{dy_p}{dt} + 2y_p(t) = e^t.$$

The second and third terms constitute the homogeneous solution. If we substitute $y_h(t) = A_1e^{-t} + A_2e^{-2t}$:

$$\frac{d^2y_h(t)}{dt^2} + 3\frac{dy_h}{dt} + 2y_h(t) = 0.$$

Thus, $y(t) = y_p(t) + y_h(t)$ and A_1 and A_2 are selected to satisfy the initial conditions.